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A regular model for cluster numbers and structure just below percolation threshold

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Abstract. A two-dimensional deterministic fractal model is proposed to imitate the geometric texture just below percolation threshold. The model is constructed for bond percolation via a rule of bond occupation on the square lattice. It shows the typical percolation behaviour as a function of a parameter p (p is the bond concentration). The model describes the approach towards the threshold below criticality. The critical bond concentration p_c and the correlation length exponent ν are found. The scaling property of the cluster size distribution is studied below the threshold. The critical concentration and critical exponents are shown to agree with those derived from the regular model proposed for the geometric texture above the threshold. The scaling relations for the critical exponents are shown to be satisfied.

1. Introduction

The geometry and statistics of clusters is one of the most important problems in phase transitions and critical phenomena. Properties of cluster numbers and structure in percolation have been reviewed in Stauffer (1979, 1985). Recently, there has been increasing interest in exact mathematical fractals (Mandelbrot 1982, Vicsek 1983, Given and Mandelbrot 1983, Ben-Avraham and Havlin 1983, Blumenfeld and Aharony 1985, Martin and Keefer 1985). The percolating infinite cluster is one of the most intensively studied random fractals (Deutscher *et al* 1983, Stauffer 1979, 1985, Stanley and Coniglio 1983, Kirkpatrick 1979, Kapitulnik and Deutscher 1984). Various geometrical models have been proposed to imitate the infinite incipient cluster at the percolation threshold. Three extreme models for the backbone of the infinite cluster have been presented, i.e. the family of Sierpinski gaskets, the 'links and nodes' model and the 'links-nodes-blobs' model (Coniglio 1982, Aharony *et al* 1984). Mandelbrot (1984a, b) and Mandelbrot and Given (1984) have also presented fractal models for percolation clusters at criticality. The Mandelbrot models possess the geometric and topological properties very close to the infinite cluster at the percolation threshold but do not describe the approach towards the threshold. Nagatani (1985) proposed the regular model to imitate the geometric texture just above the threshold. It was found that the typical percolation behaviour was successfully reproduced as a function of bond concentration p just above the threshold. The critical bond concentration, the correlation length exponent and the scaling property of the cluster distribution were found.

In this paper, we present the regular model for the geometric texture just below the threshold. The model has to be consistent with that above the threshold with respect to the critical concentration and critical exponents. For later convenience, we summarily explain the model presented for percolation above the threshold. The model is constructed by bond deletions. Bonds on the square lattice are recursively deleted via two rules. Three construction stages are shown in figure 1. Figures 1(a), (b) and (c) represent respectively the geometric textures obtained at the first, second and third stages. The crosses, triangles and (open and full) squares indicate, respectively, bonds deleted at the first, second and third stage. The system obtained at the N stages appears to be a superlattice made by nodes separated by a distance of $\xi = 3^N$, connected by quasi-linear links. When N is infinitely large, the model is self-similar (fractal) on smaller length scales than the connectedness length, but becomes a homogeneous square lattice on large length scales. The concentration p of bonds approaches the critical value $p_c = \frac{3}{5}$. The connectedness length diverges as $\xi \sim (p - p_c)^{-\nu}$ and $\nu = 0.5/(1 - \log 2/\log 3)$ ($= 1.3547\ldots$). The fractal dimension D of the infinite cluster and the fractal dimension D_b of its backbone are given by $D = \log 8/\log 3$ ($= 1.892\ldots$) and $D_b = \log 6/\log 3$ ($= 1.630\ldots$). The exponent, describing the power-law dependence on scale length L of the conductivity, is given by $t/\nu = \log \frac{11}{4}/\log 3$ ($= 0.9207\ldots$). For the cluster size distribution n_s, one arrives at the scaling form $n_s \sim s^{-\tau}\theta(1 - s/s_\xi)$ with $\tau = 1 + \log 9/\log 8$ ($= 2.056\ldots$) where $s_\xi \sim (p - p_c)^{-\nu D}$.

We shall construct the model to mimic the geometric texture just below the threshold. It is necessary that the model be possessed of characteristic properties that the critical concentration and critical exponents are consistent with those above the threshold. In § 2 the fractal models are constructed to imitate the geometric textures above and below the percolation threshold. In § 3 the fractals are reconstructed to correspond to configurations of the bonds on the square lattice. In § 4, by use of rules of bond

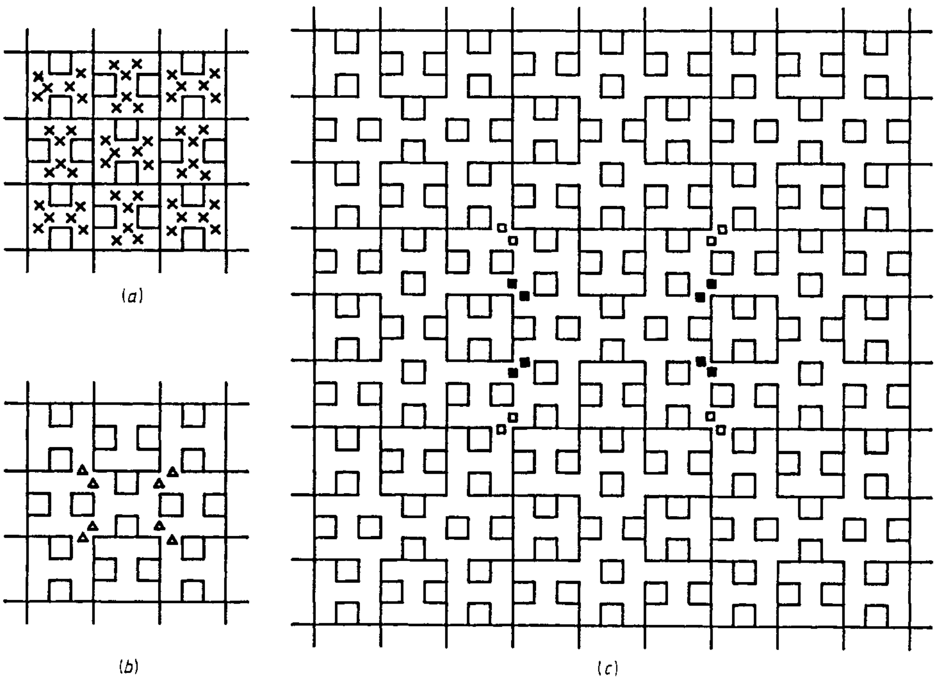


Figure 1. Three construction stages of the regular model for the geometric texture above the threshold; (a), (b) and (c) represent respectively the geometric textures, obtained at the first stage, the second stage and the third stage. The crosses, triangles and (open and full) squares indicate, respectively, bonds deleted at the first, second and third stages.

occupations, the model is presented to show the typical percolation behaviour as a function of a parameter p (p is the bond concentration). The critical concentration and the critical exponents for the scaling behaviours are derived. In § 5 the scaling relations for the critical exponents are discussed. Section 6 presents the summary.

2. Fractals above and below the threshold

We construct the fractal models to imitate the geometric textures just above and below the threshold. It is necessary that models be self-similar (fractal) on smaller length scales than the connectedness length but become homogeneous on large length scales.

In general, every lattice bond has three choices in the bond percolation: it can be empty, with probability $1-p$; it can be part of the infinite network of occupied bonds, with probability pP_∞ (P_∞ is the percolation probability) or it can be part of one of the many finite clusters, wth probability $p(1-P_\infty)$. Since each s cluster contains exactly s bonds, the probability of any lattice bond belonging to an s cluster is $P_s = sn_s$ (n_s is the number of s clusters divided by the total number of lattice bonds). The sum of all these probabilities equals unity. Then P_∞ vanishes below the threshold p_c and is non-zero above p_c. As the concentration p approaches the threshold p_c, the pair connectedness length ξ diverges, $\xi \sim |p_c - p|^{-\nu}$. The percolation probability P_∞ just above the threshold and the cluster size distribution n_s just above and just below the threshold show the following scaling behaviour (Stauffer 1979, 1985):

$$P_\infty \sim (p-p_c)^\beta \qquad \text{and} \qquad n_s \sim s^{-\tau}.$$

Fractal models, reflecting the above characteristic features of cluster numbers and structure, are constructed by use of the initiators and the generator shown by figures 2(a), (b) and (c). Figures 2(a) and (b) indicate the initiators for the geometric textures just above and just below the threshold. The length of the links (or lines) is of the order of the connectedness length ξ. The generator in figure 2(c) has the disconnected portion, compared with the initiator for the infinite cluster presented by Mandelbrot and Given (1984) (see figure 2(d)). The generator includes a connected line of eight links, joining the endpoints of the interval. This portion is called the 'coastline generator' in accord with the Mandelbrot-Koch curve. The remaining one link forms a portion that seeds new islands and is called the 'island generator'. This splits into pieces to build up finite clusters for percolation. The first two construction stages of the fractals are respectively indicated by figures 3(a) and (b) for the geometric textures above and below the threshold. Above the threshold, the system obtained at the N stages appears to be a superlattice made by nodes separated by a distance of $\xi = 3^N$,

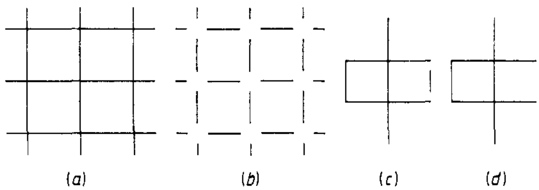


Figure 2. Initiators and generators of fractals for the geometric textures in percolation; (a) and (b) indicate the initiators for geometries just above and below the threshold; (c) shows the generator for the geometric textures; (d) represents the Mandelbrot-Koch curve, being the generator for infinite cluster.

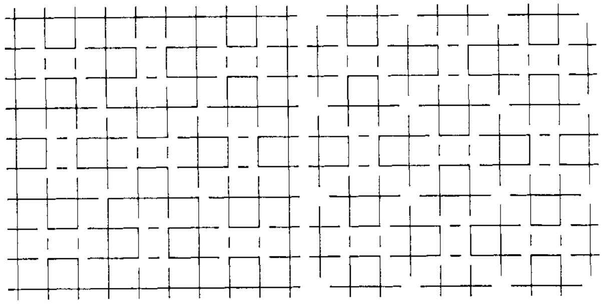


Figure 3. The first two construction stages of the fractals for the geometric textures (*a*) above and (*b*) below the threshold.

connected by quasi-linear links. We obtain the square lattice with self-similar structures on smaller length scales than the connectedness length $\xi = 3^N$. This corresponds to the percolating network. Islands, separated from the percolating network, correspond to finite clusters. The fractal dimension D of the infinite cluster and the fractal dimension D_b of its backbone are respectively given by $D = \log 8/\log 3$ and $D_b = \log 6/\log 3$. The cluster size distribution consists of a sum of delta functions:

$$n_s \sim \sum_{k=1}^{\infty} (\tfrac{1}{9})^k \delta(s - (3^D)^k)\theta(1 - s/(3^D)^N). \tag{1}$$

By spreading the delta functions over the interval and taking into account that $s \sim (3^D)^k$, one can arrive at the scaling form

$$n_s \sim s^{-\tau}\theta(1 - s/s_\xi) \tag{2}$$

with $\tau = 1 + \log 9/\log 8 = 1 + d/D$ where $s_\xi \sim (3^D)^N$.

3. Geometry as a function of the bond concentration

We reconstruct the fractals to correspond to configurations of the bonds on the square lattice. The models are presented to show the typical percolation behaviour as a function of the bond concentration p. For the geometric texture just above the threshold, bonds on the square lattice are recursively deleted via a rule. On the other hand, for the geometric texture just below the threshold, bonds are recursively occupied via a rule. The models, constructed by the bond deletions and bond occupations, present the geometric textures as a function of the bond concentration p. Thus, one can describe the approaches towards the threshold. One can obtain the regular models to show the typical percolation behaviour as a function of a parameter p. The first three stages of constructions, for the geometric textures above and below the threshold, are shown by figure 1 (where the bonds indicated by full squares are occupied) and figure 4. Figure 1 has been explained in the introduction, except that the bonds marked by full squares are occupied. The concentration of bonds occupied after the Nth stage via the first rule of bond deletions, is given by

$$p(N)(>p_c) = 1 - \tfrac{1}{3} - \sum_{n=2}^{N} 4/9^n. \tag{3}$$

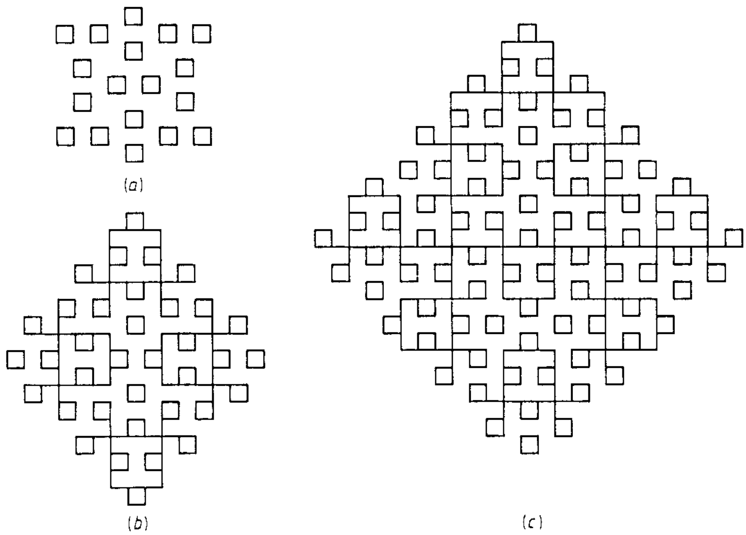


Figure 4. The correspondence of the fractal geometry, shown by figure 3(*b*), with configurations of the bonds on the square lattice; (*a*), (*b*) and (*c*) indicate respectively the geometric textures below the threshold, obtained at the first, second and third stages of construction. They show the geometry as a function of the bond concentration.

When N is infinitely large, the concentration p approaches the critical value p_c:

$$p_c = \lim_{N\to\infty} p(N) = \tfrac{11}{18} (= 0.611\ldots). \tag{4}$$

The connectedness length diverges as

$$\xi \sim (p - p_c)^{-1/2}. \tag{5}$$

Figures 4(*a*), (*b*) and (*c*) indicate respectively the geometric textures below the threshold obtained at the first, second and third stages of construction. The concentration of bonds after the Nth stage is given by

$$p(N)(< p_c) = \tfrac{4}{9} + \sum_{n=2}^{N} 12/9^n. \tag{6}$$

The critical concentration and the connectedness length are obtained from the approach below the threshold:

$$p_c = \tfrac{11}{18} \qquad \text{and} \qquad \xi \sim (p_c - p)^{-1/2}. \tag{7}$$

The critical concentration p_c and the correlation length exponent ν have the same values above and below the threshold. Thus we obtain the regular models as a function of bond concentration p to describe the approaches towards the threshold above and below p_c.

These models are a poor approximation for the correlation length exponent. In order to improve the exponent ν, we introduced the second rule of bond deletions for the geometric texture above the threshold. The second rule was summarised as follows: bonds into the islands separated from the percolating network are furthermore deleted recursively (Nagatani 1985). The full squares in figure 1 represent bonds deleted at the third stage by the second rule of bond deletions. One can obtain the improved values

$p_c = \frac{3}{5}$ ($= 0.6$) and $\nu = 0.5/(1-\log 2/\log 3)$ ($= 1.3547\ldots$). We shall construct the model below the threshold to be possessed of characteristic properties such that the critical concentration and critical exponents are consistent with those above the threshold. In the following section the model for the geometric texture below the threshold is constructed via the two rules of bond occupations.

4. The regular model just below the threshold

Now we try to imitate bond percolation with the help of a regular construction. The regular model is constructed by the following bond occupations. Bonds on the square lattice are recursively occupied via two rules. First we apply the first rule of the bond occupation. Bonds are occupied at the Nth stage such that squares with the edges of 3^{N-1} length are periodically constructed with the period 2×3^N. The positions (i, j) of the square-centres satisfy the relations

$$\begin{aligned}
&\cos(\pi i/3^N - \pi/2) = \cos(\pi j/3^N - \pi/6) = 1\\
&\cos(\pi i/3^N - \pi/2) = \cos(\pi j/3^N - 5\pi/6) = 1\\
&\cos(\pi i/3^N + \pi/6) = \cos(\pi j/3^N - \pi/2) = 1\\
&\cos(\pi i/3^N + 5\pi/6) = \cos(\pi j/3^N - \pi/2) = 1\\
&\cos(\pi i/3^N + \pi/2) = \cos(\pi j/3^N + \pi/6) = 1\\
&\cos(\pi i/3^N + \pi/2) = \cos(\pi j/3^N + 5\pi/6) = 1\\
&\cos(\pi i/3^N - \pi/6) = \cos(\pi j/3^N + \pi/2) = 1\\
&\cos(\pi i/3^N - 5\pi/6) = \cos(\pi j/3^N + \pi/2) = 1.
\end{aligned} \tag{8}$$

Figure 5 indicates the configuration of the squares within the unit period. Secondly, we apply the second rule of bond occupation to the resultant lattice. This rule works

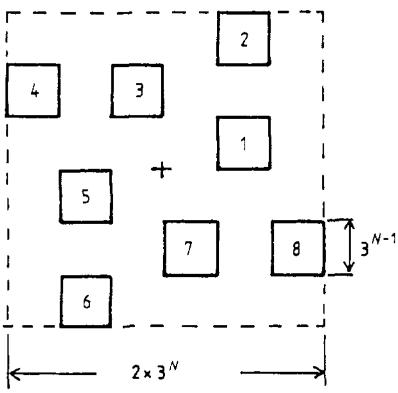
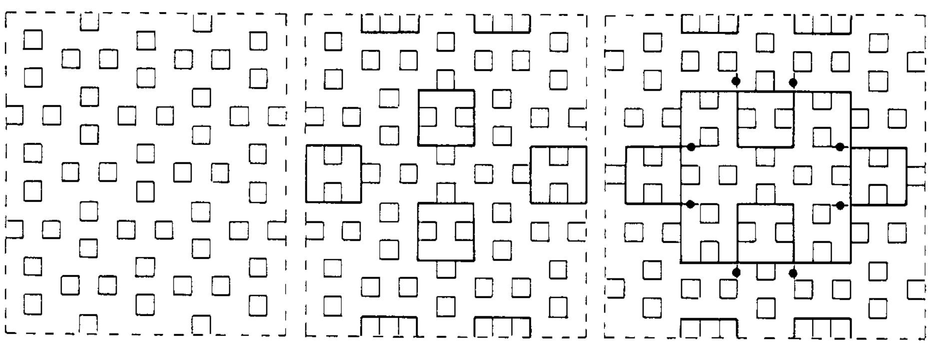


Figure 5. Bond occupations at the Nth stage by the first rule. Squares with the edges of 3^{N-1} length are periodically constructed with the period 2×3^N. The configuration of the squares within the unit period is represented. When the origin is placed at the point indicated by the cross, the positions (i, j) of the square-centres are respectively given by (1): $(3^N/2, 3^{N-1}/2)$, (2): $(3^N/2, (5\times 3^{N-1}/2))$, (3): $(-3^{N-1}/2, 3^N/2)$, (4): $(-5\times 3^{N-1}/2, 3^N/2)$, (5): $(-3^N/2, -3^{N-1}/2)$, (6): $(-3^N/2, (-5\times 3^{N-1}/2))$, (7): $(3^{N-1}/2, -3^N/2)$ and (8): $(5\times 3^{N-1}/2, -3^N/2)$.

at stages larger than $N = 2$. The second rule is summarised as follows: bonds are occupied such that the internal structure of the resultant squares, constructed by the first rule, is self-similar with the sufficiently large N. Three construction stages of our regular model are shown in figure 6. Figures 6(a), (b) and (c) represent the lattices constructed at the first, second and third stages respectively. The full circles in figure 6(c) indicate bonds occupied by the second rule at the third stage ($N = 3$). Figure 7 shows the construction of a finite cluster from the square with edges of the 3^3 length by means of the second rule of bond occupation at the fourth stage ($N = 4$). Bonds marked by full circles indicate those occupied by the second rule where a part (one edge) of the square is shown. In this manner, finite clusters are generated by use of the rules of bond occupation. The finite clusters are shown in figure 8. It is found that a large cluster is the fractal with the initiator of square and the generator of the Mandelbrot-Koch curve. The concentration $c(N)$ of bonds, occupied at the Nth stage



(*a*) (*b*) (*c*)

Figure 6. Three construction stages of the regular model; (a), (b) and (c) represent the lattices constructed at the first, second and third stages respectively. The full circles in (c) indicate bonds occupied by the second rule.

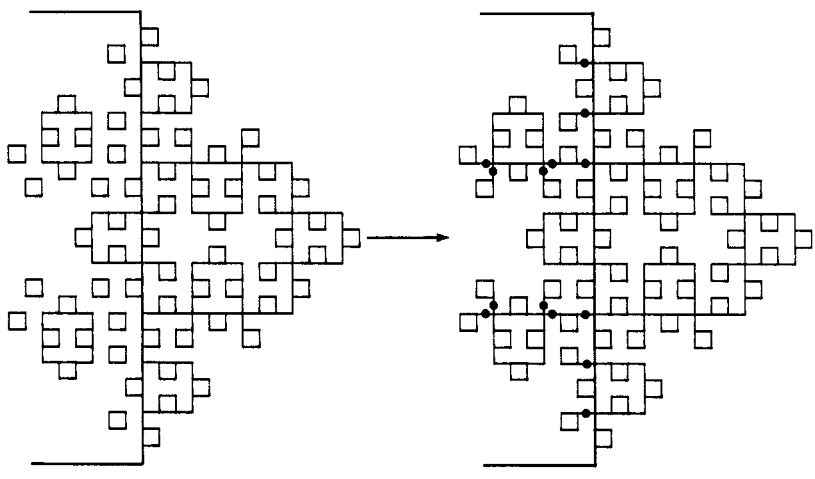


Figure 7. The construction of a finite cluster from the square with edges of the 3^3 length by means of the second rule of bond occupation at the fourth stage ($N = 4$). Bonds marked by full circles indicate those occupied by the second rule where a part (one edge) of the square is shown. The finite cluster on the right-hand side generates.

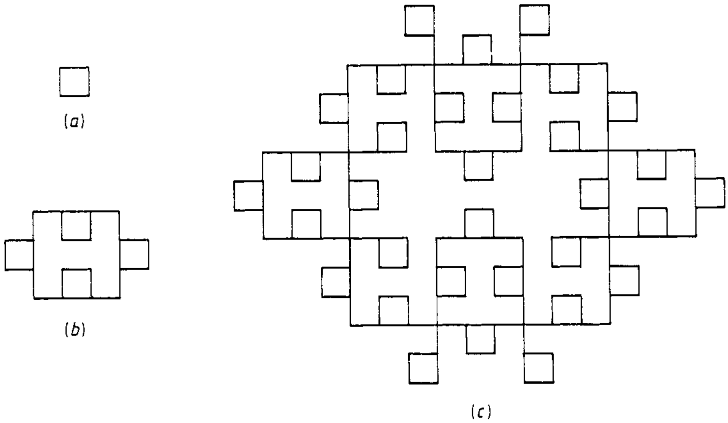


Figure 8. The finite clusters generated by the bond occupation. Finite clusters shown in (a), (b) and (c) represent, respectively, those generated at the first, second and third stages. A large cluster is the fractal with the initiator of square and generator of the Mandelbrot-Koch curve.

via the two rules of bond occupation, is given by

$$c(N) = 8/9^N + (4/9^N)(4^{N-1}/3 - \tfrac{4}{3}) \tag{9}$$

where $c(1) = \frac{4}{9}$ and $c(2) = 8/9^2$.

The concentration $p(N)$ of bonds after N stages is given by

$$p(N) = \sum_{n=1}^{N} c(n). \tag{10}$$

When N is infinitely large, the concentration p approaches the critical value p_c:

$$p_c = \lim_{N\to\infty} p(N) = \tfrac{3}{5}(=0.6). \tag{11}$$

The pair connectedness length ξ is given by

$$\xi \sim 3^{N-1}. \tag{12}$$

We obtain

$$\delta p(\equiv p_c - p(N)) \sim (\tfrac{4}{9})^N \sim \xi^{-2(1-\log 2/\log 3)}. \tag{13}$$

The connectedness length diverges as

$$\xi \sim (p_c - p)^{-\nu} \qquad \text{and} \qquad \nu = 0.5/(1-\log 2/\log 3)(=1.3547\ldots). \tag{14}$$

The critical concentration p_c and the correlation length exponent ν have the same values above and below the threshold. Thus we obtain the regular model below p_c, consistent with that above p_c. The value for the correlation length exponent agrees with that derived in a completely different fashion by Klein *et al* (1978) and was then thought to be perhaps exact. In order to obtain the cluster size distribution, one should note that the largest cluster generated in the kth stage of the process of bond occupations contains $s(k) \sim 8^k$ bonds. For $s < (3^D)^N$ the cluster size distribution consists of a sum of delta functions:

$$n_s \sim \sum_{k=1}^{\infty} (\tfrac{1}{9})^k \left(1 - \sum_{n=1}^{N-k} \tfrac{1}{9}(\tfrac{4}{9})^n\right) \delta(s - (3^D)^k)\,\theta(1 - s/(3^D)^N). \tag{15}$$

Table 1. List of the physical and geometric properties determined analytically by our regular model, compared with other sources.

p_c	ν	τ	σ	D	D_b
$\frac{3}{5}$	$0.5/(1-\log_3 2)$	$1+\log_8 9$	$(1-\log_3 2)/0.5\log_3 8$	$\log_3 8$	$\log_3 6$
(0.6)	(1.354)	(2.056)	(0.389)	(1.892)	(1.630)
0.5[a]	$\frac{4}{3}$[a]	2.05[a]	0.39[a]	1.90[b] $\frac{91}{48}$[a]	1.62[c]

[a] Stauffer (1985).
[b] Kapitulnik and Deutscher (1984).
[c] Herrmann and Stanley (1984).

By spreading the delta functions over the interval we obtain

$$n_s \sim s^{-\tau}\theta(1-s/s_\xi) \tag{16}$$

with $\tau = 1+\log 9/\log 8$ where $s_\xi \sim (3^D)^N \sim (p_c - p)^{-\nu D}$, so $1/\sigma = \nu D$ is obtained.

Table 1 lists the geometric and physical properties, determined analytically by our regular model. In table 1, the second line shows estimated scaling exponents for the two-dimensional (random) percolation to compare our results more completely with those of random percolation. The scaling exponents obtained below the threshold are in agreement with those above the threshold (Nagatani 1985).

5. Scaling relations

We will show that all the conventional scaling relations between the critical exponents are satisfied in this construction. Our scaling form for the cluster numbers (16) is consistent with the conventional scaling assumption

$$n_s(p) = s^{-\tau} f\{(p-p_c)s^\sigma\}. \tag{17}$$

The scaling relations between critical exponents, derived from the scaling form (17), are satisfied with our model:

$$\beta = (\tau-2)/\sigma \qquad \gamma = (3-\tau)/\sigma \qquad (2-\alpha) = (\tau-1)/\sigma. \tag{18}$$

We obtain the scaling form for the radius of cluster (Stauffer 1985):

$$R_s \sim s^\rho \theta\{c+(p-p_c)s^\sigma\} \tag{19}$$

where $\rho = 1/D$.

The following scaling relations are derived from (19):

$$\rho = \sigma\nu \qquad \text{and} \qquad d\nu = (\tau-1)/\sigma. \tag{20}$$

All the conventional scaling relations between the critical exponents for cluster numbers and structure are satisfied in this model.

6. Summary

We summarise that the geometric texture just below percolation threshold can be imitated by the regular model. The most important feature of the regular model is that it is possible to obtain explicit expressions for the quantities characterising the approach towards the percolation threshold. The model shows the typical percolation behaviour as a function of the bond concentration p. The other important feature of the regular model is that it is possible to obtain explicit expressions for the quantities characterising the statistics of clusters defined in percolation. The regular construction of percolation, simulating the scaling properties, is shown to be possessed of characteristic properties of cluster numbers and structure below the threshold. All the conventional scaling relations between critical exponents are satisfied in this construction. Our critical exponents are very close to the exact ones.

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